Física Matemática - Prova 2

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Início: 14:00 - duração: 3:00 horas

Só serão consideradas as respostas que forem devidamente justificadas.
Não é permitido o uso de calculadoras.

Questão 01: Série de Fourier

Seja f uma função definida por

$$f(x) = \begin{cases} 1+x & -\pi < x \leq 0 \\ 1-x & 0 < x \leq \pi \\ 0 & \text{caso contrário} \end{cases}$$

(a) **(0,5)** Esboce o gráfico de $f(x)$ em função de x.
(b) **(2,5)** Encontre a série de Fourier de $f(x)$ no intervalo $[-\pi, \pi]$.

Questão 02: (4,0) Transformada de Fourier

Calcule a transformada de Fourier da lorentziana dada por

$$f(x) = \frac{1}{1+\alpha x^2}, \quad \alpha > 0.$$

Dica: Estenda a integração para o plano complexo e faça uso do teorema integral de Cauchy e do lema de Jordan.

Questão 03: Função gama

Uma das definições da função gama, conforme visto em sala de aula, é a integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt.$$

Usando essa definição, obtenha as seguintes propriedades:

(a) **(1,0)** $\Gamma(z) = (z-1)\Gamma(z-1)$. *Dica*: Use integração por partes.

(b) **(1,0)** $\Gamma(1/2) = \sqrt{\pi}$. *Dica*: Use substituição de variáveis.

(c) **(1,0)** $(3/2)! = \frac{3}{4}\sqrt{\pi}$. *Dica*: Use os resultados dos itens (a) e (b).

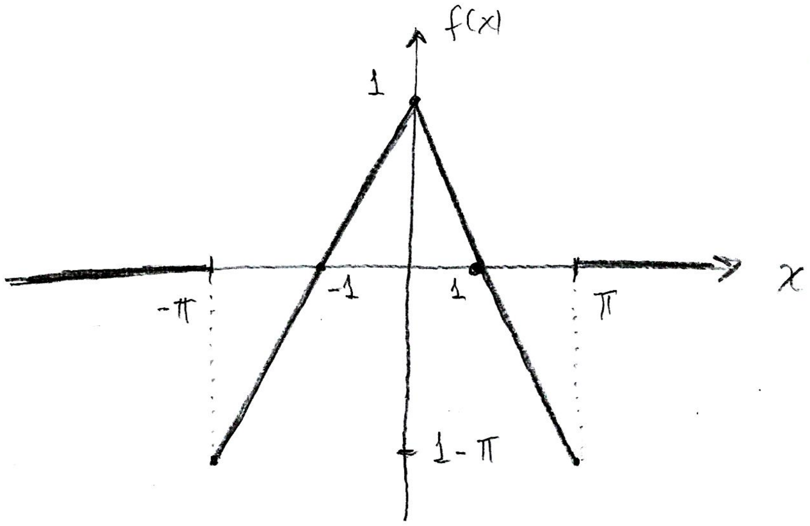
Dado: $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$.

GABARITO

1.

$$f(x) = \begin{cases} 1+x, & -\pi < x \leq 0 \\ 1-x, & 0 < x \leq \pi \\ 0, & \text{c.c.} \end{cases}$$

A)



B) f É PAR $\Rightarrow$ $b_n = 0$.

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} 2f(x)\,dx \quad (f \text{ É PAR})$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(1-x)\,dx = \frac{2}{\pi}\left(x - \frac{x^2}{2}\right)\Big|_0^{\pi} = \frac{2}{\pi}\left(\pi - \frac{\pi^2}{2}\right)$$

$$\Rightarrow \boxed{a_0 = 2 - \pi}$$

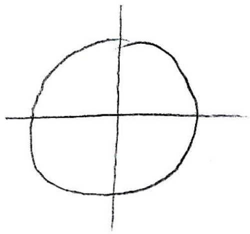
$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(1-x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi}\cos nx\,dx - \int_0^{\pi} x\cos nx\,dx\right]$$

$$\int_0^{\pi} \cos nx\,dx = \frac{\operatorname{sen} nx}{n}\Bigg|_0^{\pi} = \frac{1}{n}\left(\operatorname{sen} n\pi - \operatorname{sen} 0\right)$$

$$= 0$$



$$\int_0^{\pi} x \cos nx\,dx = uv - \int v\,du$$

$u = x$

$dv = \cos nx\,dx$

$du = dx$

$v = \dfrac{\operatorname{sen} nx}{n}$

$$= \frac{x}{n}\operatorname{sen} nx\Bigg|_0^{\pi} - \frac{1}{n}\int_0^{\pi} \operatorname{sen} nx\,dx$$

$$= 0 + \frac{1}{n^2}\cos nx\Bigg|_0^{\pi}$$

$$= \frac{1}{n^2}\left(\cos n\pi - 1\right) = \frac{(-1)^n - 1}{n^2} \Rightarrow$$

$$a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right] \Rightarrow$$

$$f(x) = \frac{2-\pi}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos nx$$

OU

$$\boxed{f(x) = 1 - \frac{\pi}{2} + \frac{4}{\pi}\sum_{\substack{n=1 \\ n \text{ ÍMPAR}}}^{\infty} \frac{\cos nx}{n^2}} \qquad -\pi < x < \pi.$$

2 - $f(x) = \dfrac{1}{1+\alpha x^2}$, $\alpha > 0$.

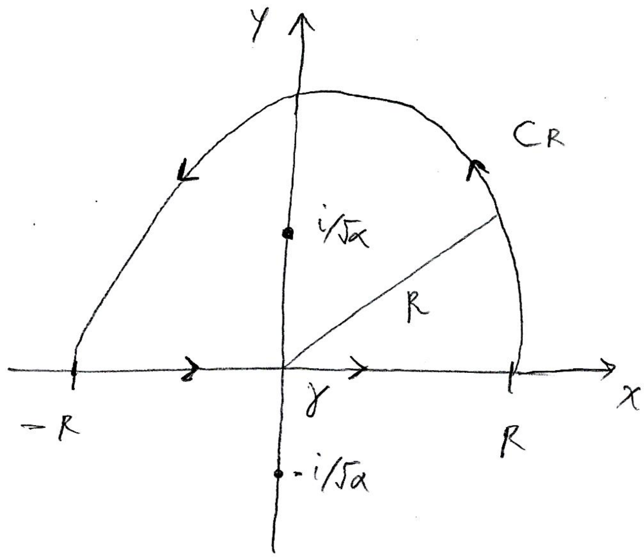
$$\tilde{f}(k) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{1+\alpha x^2}\, dx$$

POLOS: $1+\alpha x^2 = 0$

$x^2 = -1/\alpha$

ESTENDENDO PARA O PLANO COMPLEXO $(k>0)$: $\Rightarrow \begin{cases} x_1 = i/\sqrt{\alpha} \\ x_2 = -i/\sqrt{\alpha} \end{cases}$

$$\oint_C \frac{e^{ikz}\,dz}{1+\alpha z^2} = \int_{C_R} \frac{e^{ikz}\,dz}{1+\alpha z^2} + \int_\gamma \frac{e^{ikz}\,dz}{1+\alpha z^2}$$



NO LIMITE $R \to \infty$, TEMOS

$$\oint_C \frac{e^{ikz}\,dz}{1+\alpha z^2} = \int_{C_R} \frac{e^{ikz}\,dz}{1+\alpha z^2} + \int_{-\infty}^{\infty} \frac{e^{ikx}\,dx}{1+\alpha x^2}$$

$\Rightarrow$

$$\int_{-\infty}^{\infty} \frac{e^{ikx}\,dx}{1+\alpha x^2} = \oint_C \frac{e^{ikz}\,dz}{1+\alpha z^2} - \int_{C_R} \frac{e^{ikz}\,dz}{1+\alpha z^2}$$

$$\oint \frac{e^{ikz}\,dz}{\alpha(z - i/\sqrt{\alpha})(z+i/\sqrt{\alpha})} = 2\pi i\, g(i/\sqrt{\alpha}) = 2\pi i \cdot \frac{e^{ik \cdot i/\sqrt{\alpha}}}{\alpha(2i/\sqrt{\alpha})}$$

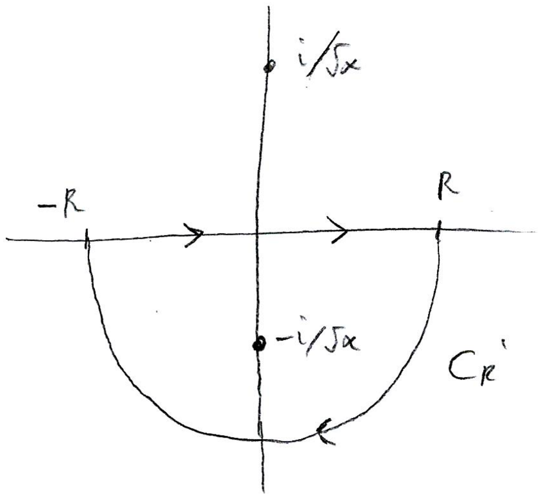
$$= \frac{\pi}{\sqrt{\alpha}} e^{-k/\sqrt{\alpha}}$$

$$\int_{C_R} \frac{e^{ikz}}{1+\alpha z^2}\,dz = 0 \quad \text{SE} \begin{cases} R \to \infty \\ k > 0 \end{cases} \quad \text{(JORDAN)}$$

ENTÃO $$\tilde{f}(k) = \frac{\pi}{\sqrt{\alpha}}\, e^{-k/\sqrt{\alpha}}\, H(k) \qquad (k>0)$$

SE $k<0$, A INTEGRAÇÃO DEVE SER POR BAIXO PARA O LEMA DE JORDAN SER ÚTIL.

$R \to \infty$:



$$\oint_{C'} \frac{e^{ikz}\,dz}{1+\alpha z^2} =$$

$$= \int_{-\infty}^{\infty} \frac{e^{ikx}\,dx}{1+\alpha x^2} + \underbrace{\int_{C_R'} \frac{e^{ikz}\,dz}{1+\alpha z^2}}_{0 \text{ (LEMA DE JORDAN)}} \Rightarrow$$

$$\int_{-\infty}^{\infty} \frac{e^{ikx}\,dx}{1+\alpha x^2} = \oint_{C'} \frac{e^{ikz}\,dz}{\alpha(z-i/\sqrt{\alpha})(z+i/\sqrt{\alpha})}$$

$$\oint_{C'} \frac{e^{ikz}\,dz}{\alpha(z-i/\sqrt{\alpha})(z+i/\sqrt{\alpha})} = -2\pi i\, g(-i/\sqrt{\alpha})$$

$$= -2\pi i \cdot \frac{e^{ik\cdot(-i)/\sqrt{\alpha}}}{\alpha\cdot(-2i)/\sqrt{\alpha}} = \frac{\pi}{\sqrt{\alpha}}\, e^{k/\sqrt{\alpha}} \qquad (k<0)$$

$\Rightarrow$

$$\boxed{\tilde{f}(k) = \frac{\pi}{\sqrt{\alpha}}\left[e^{-k/\sqrt{\alpha}} H(k) + e^{k/\sqrt{\alpha}} H(-k) \right]}$$

OU

$$\boxed{\tilde{f}(k) = \frac{\pi}{\sqrt{\alpha}}\, e^{-|k|/\sqrt{\alpha}}}$$

3. $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$

A) $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt = uv - \int v\, du$

$$\begin{cases} u = t^{z-1} \\ dv = e^{-t} dt \\ du = (z-1) t^{z-2} dt \\ v = -e^{-t} \end{cases} \Rightarrow$$

$$\int_0^\infty e^{-t} t^{z-1} dt = \underbrace{-e^{-t} t^{z-1} \Big|_0^\infty}_{=0} + \int_0^\infty e^{-t} (z-1) t^{z-2} dt$$

$$\Rightarrow \quad \Gamma(z) = (z-1) \int_0^\infty e^{-t} t^{z-2} dt$$

$$\boxed{\Gamma(z) = (z-1)\, \Gamma(z-1)}$$

B) $\Gamma(1/2) = \int_0^\infty e^{-t} t^{-1/2} dt$

$t = y^2 \Rightarrow$

$dt = 2y\, dy \Rightarrow$

$$\int_0^\infty e^{-t} t^{-1/2} dt = \int_0^\infty e^{-y^2} \cdot y^{-1} \cdot 2y\, dy$$

$$= 2 \int_0^\infty e^{-y^2} dy = \int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$$

$$\Rightarrow \quad \boxed{\Gamma(1/2) = \sqrt{\pi}}$$

c) $\left(\frac{3}{2}\right)! = ?$

$\Gamma(z+1) = z! \Rightarrow$

$(3/2)! = \Gamma(5/2) = 3/2\ \Gamma(3/2)$

$= 3/2 \cdot 1/2\ \Gamma(1/2)$

$= \frac{3}{4}\sqrt{\pi} \Rightarrow$

$$\boxed{(3/2)! = \frac{3}{4}\sqrt{\pi}}$$